

The gravitational field of a global monopole

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Abstract

We present an exact solution to the non-linear equation which describes a global monopole in the flat space. We re-examine the metric and the geodesics outside the global monopole. We will see that a global monopole produces a repulsive gravitational field outside the core in addition to a solid angular deficit. The lensing property of the global monopole and the global monopole-antimonopole annihilation mechanism are studied.

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1 Introduction

The idea that magnetic monopoles ought to exist has proved to be remarkably durable. A persuasive argument was first put forward by Dirac [1]. Many years later, another very good argument emerged. The 't Hooft-Polyakov monopole solution is a synthesis of the Dirac monopole in the non-Abelian gauge theories [2],[3]. The monopole solutions of the realistic grand unified models based on the gauge groups $SU(5)$ and $SO(10)$ have been considered in [4] and [5]. An extensive list of the Kaluza-Klein monopole literatures was made in [6] – [11]. Surprising and qualitative phenomena arise when one considers the quantum mechanics of electrically charged fermions interacting with magnetic monopoles [12] – [16].

Recently, Barriola and Vilenkin [17] have shown an approximate solution of the Einstein equations for the metric outside a global monopole, resulting from a global symmetry breaking. Such a monopole has Goldstone fields with energy density decreasing with the distance only as r^{-2} , so that the total energy is linearly divergent at large distance. Neglecting the mass term, the monopole metric describes a space with a deficit solid angle. The area of a sphere of radius r is not $4\pi r^2$, but $4\pi(1 - 8\pi G\eta^2)r^2$. Requiring that the mass density in such a monopole should not greatly exceed the critical density implies there is at most one global monopole in the local group of galaxies. Equally stringent bounds are also derived which do not depend on cosmological assumptions, using the large tidal gravitational forces associated with the global monopole [18].

In this paper, we present an exact solution to the non-linear equation which describes the global monopole in the flat space. We re-examine the metric outside a global monopole. We will see a repulsive gravitational field outside the core in addition to a solid angular deficit. We examine the geodesics and find that the deflected angle is small. For ultra-relativistic particles, the deflected

angle also depends upon the impact parameter. If the impact parameter is of a galaxy scale and δ is of the grand unification scale, the term with impact parameter can be neglected. However, if the term is of the scale of a mini soliton star [19], this term is important. The lensing property of the global monopole, as well as other classical effects, are studied. We show that the repulsive gravitation force between global monopole and antimonopole ($M\overline{M}$) does not change the $M\overline{M}$ annihilation proposed by Barriola and Vilenkin [17].

2 An exact solution of the global monopole in flat space

The simplest model that gives rise to the global monopole is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{4}\lambda(\phi^a\phi^a - \eta^2)^2 \quad (1)$$

where ϕ^a is a triplet of scalar fields, $a = 1, 2, 3$. The model has a global $O(3)$ symmetry which is spontaneously broken to $U(1)$. The field configuration describing a monopole is

$$\phi^a = \eta f(r) \frac{x^a}{r} \quad (2)$$

The field equations for ϕ^a in the flat metric reduce to a single equation for $f(r)$

$$f'' + \frac{2}{r}f' - \frac{2}{r^2}f - \frac{f(f^2 - 1)}{\delta^2} = 0 \quad (3)$$

where $\delta = (\eta\sqrt{\lambda})^{-1}$ is the core radius of the monopole. The function $f(r)$ grows linearly when $r < \delta$ and exponentially approaches unity as soon as $r \geq \delta$. Barriola and Vilenkin [17] took $f = 1$ outside the core which is an approximation to the exact solution.

On the other hand, Lan and Wang [20] have discussed an exact soliton solution of the ice-like structure in the form of a series of hyperbolic functions. In similar ways, we will give an exact solution of the global monopole. Furthermore, we also discuss the convergence of these series solutions.

Setting $x = r/\delta$ in equation (3), we obtain

$$x^2 f'' + 2x f' - x^2 f^3 + (x^2 - 2)f = 0 \quad (4)$$

which satisfies the boundary conditions

$$f(0) = 0, \quad (5)$$

$$f(\infty) = 1. \quad (6)$$

This problem can be solved explicitly by the method of hyperbolic functions, in the form

$$f(x) = \sum_{n=0}^{\infty} c_n \tanh^{2n+1} \frac{x}{\sqrt{2}} \quad (7)$$

which satisfies the boundary condition at $x \rightarrow \infty$

$$\sum_{n=0}^{\infty} c_n = 1. \quad (8)$$

By using the formula

$$x = \sqrt{2} \tanh \frac{x}{\sqrt{2}} \sum_{n=0}^{\infty} \tanh^{2n} \frac{x}{\sqrt{2}} (2n+1)^{-1}, \quad (9)$$

we have

$$x^2 = 2 \tanh^2 \frac{x}{\sqrt{2}} \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{l=0}^{\infty} \frac{1}{2l+1} \right) \tanh^{2n} \frac{x}{\sqrt{2}}. \quad (10)$$

For the determination of c_n , we first substitute equations (7), (8) and (10) into equation (4). We show that the recursion formula of the coefficients c_n can be expressed as follows:

$$\begin{aligned} n(2n+3)c_n &= \sum_{l+k=n-1} \left(\frac{2(2k+1)}{(2l+1)(2l+3)} + 4k(k+1)d_l - (2k+1)kd_{l-1} \right. \\ &\quad \left. - (k+1)(2k+1)d_{l-1} \right) c_k + \sum_{i+j+k+l=n-2} d_l c_i c_j c_k, \end{aligned} \quad (11)$$

where

$$d_l = \begin{cases} 0 & \text{if } l = -1 \\ \frac{1}{l+1} \sum_{i=0}^l \frac{1}{2i+1} & \text{if } l = 0, 1, 2, 3, \dots \end{cases} \quad (12)$$

Equations (8) and (11) lead to a simple method to determine c_n .

By using equation (8), we have $|f(x)| < 1$ for $x \geq 0$. The series of hyperbolic functions (7) converges uniformly in the region $x \geq 0$. Thus we believe that the function series (7) shows an exact solution of equation (4).

Next, we consider the approximate solution of N th order

$$f^{(N)} = \sum_{n=0}^N c_n^{(N)} \tanh^{2n+1} \frac{x}{\sqrt{2}} \quad (13)$$

which satisfies the infinite boundary condition,

$$\sum_{n=0}^N c_n^{(N)} = 1. \quad (14)$$

The approximate solution of zero order is

$$f^{(0)} = \tanh \frac{x}{\sqrt{2}}. \quad (15)$$

The approximate solution of 10th order is

$$\begin{aligned}
f^{10}(x) = & 0.7539 \tanh \frac{x}{\sqrt{2}} + 0.1005 \tanh^3 \frac{x}{\sqrt{2}} + 0.0414 \tanh^5 \frac{x}{\sqrt{2}} + \\
& + 0.0249 \tanh^7 \frac{x}{\sqrt{2}} + 0.0178 \tanh^9 \frac{x}{\sqrt{2}} + 0.0141 \tanh^{11} \frac{x}{\sqrt{2}} + \\
& + 0.0118 \tanh^{13} \frac{x}{\sqrt{2}} + 0.0103 \tanh^{15} \frac{x}{\sqrt{2}} + 0.0092 \tanh^{17} \frac{x}{\sqrt{2}} + \\
& + 0.0084 \tanh^{19} \frac{x}{\sqrt{2}} + 0.0078 \tanh^{21} \frac{x}{\sqrt{2}}
\end{aligned} \tag{16}$$

For $N \geq 10$, the relevant error is less than 0.5%. When N increases, the relevant error will decrease. One can in principle reach an arbitrary accuracy if one preserves sufficient terms in equations (13) and (14) .

In the region $x \gg 1$, the solution (7) can be written as

$$f(x) = 1 - be^{-\sqrt{2}x} + o(e^{-\sqrt{2}x}) \tag{17}$$

where

$$b = \sum_{n=0}^{\infty} 2(2n+1)c_n = \lim_{N \rightarrow \infty} \sum_{n=0}^N 2(2n+1)c_n^{(N)}. \tag{18}$$

3 The metric around a global monopole

The most general static metric with spherical symmetry can be written as

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{19}$$

The non-vanishing components of the Ricci tensor for this metric are

$$R_{tt} = -\frac{B''}{2A} + \frac{B'}{4A} \left(\frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \frac{B'}{A}, \tag{20}$$

$$R_{rr} = \frac{B''}{2B} - \frac{B'}{4B} \left(\frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \frac{A'}{A}, \tag{21}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A} \left(-\frac{A'}{A} + \frac{B'}{B} \right) + \frac{1}{A}, \tag{22}$$

$$R_{\varphi\varphi} = \sin^2\theta R_{\theta\theta}. \tag{23}$$

The energy-momentum tensor of the monopole is given by

$$T^t_t = \eta^2 \frac{f'^2}{2A} + \eta^2 \frac{f^2}{r^2} + \frac{1}{4} \lambda \eta^4 (f^2 - 1)^2, \tag{24}$$

$$T^r_r = -\eta^2 \frac{f'^2}{2A} + \eta^2 \frac{f^2}{r^2} + \frac{1}{4} \lambda \eta^4 (f^2 - 1)^2, \tag{25}$$

$$T^\theta_\theta = T^\varphi_\varphi = \eta^2 \frac{f'^2}{2A} + \frac{1}{4} \lambda \eta^4 (f^2 - 1)^2. \tag{26}$$

In the flat space the monopole core has size $\delta \sim (\eta\sqrt{\lambda})^{-1}$. For $\eta \ll m_p$, where m_p is the planck mass, we expect that gravity does not substantially change the structure of the monopole at small distance so that the flat space estimation of δ still applies.

The useful Einstein equations are given by

$$\frac{1}{A} \left(\frac{1}{r^2} - \frac{1}{r} \frac{A'}{A} \right) - \frac{1}{r^2} = 8\pi G T^t{}_t, \quad (27)$$

$$\frac{1}{A} \left(\frac{1}{r^2} + \frac{1}{r} \frac{B'}{B} \right) - \frac{1}{r^2} = 8\pi G T^r{}_r. \quad (28)$$

From equation (27), we get the general relation for $A(r)$,

$$A^{-1}(r) = 1 - \frac{8\pi G}{r} \int_0^r T^t{}_t r^2 dr \quad (29)$$

and from equation (28), we get the general relation for $B(r)$,

$$B(r) = A^{-1}(r) \exp(-8\mu \int_0^r f^2 r dr). \quad (30)$$

Substituting the solution $f(r)$ into (29) and (30), we get the solutions for $A(r)$ and $B(r)$. In the linear approximation, i.e, assuming that both $A(r)$ and $B(r)$ are very close to unity, the result is

$$A(r) = 1 + 8\mu + \frac{Gm}{r}, \quad (31)$$

$$B(r) = 1 - 8(1 + \beta)\mu - \frac{Gm}{r}. \quad (32)$$

where $\mu = \pi G \eta^2$, and

$$m = 8\pi \eta^2 \int_0^\infty \left(f'^2 + \frac{f^2 - 1}{r^2} + \frac{1}{4} \lambda \eta^2 (f^2 - 1)^2 \right) r^2 dr \quad (33)$$

$$\beta = \int_0^\infty f'^2 dr. \quad (34)$$

By using equation (16), we obtain the numerical result for m ,

$$m = -46.99 \eta^2 \delta. \quad (35)$$

In flat space, the core of a global monopole has size $\delta \sim \lambda^{-\frac{1}{2}} \eta^{-1}$. We expect that gravity does not notably change the structure of the monopole at small distance, so that,

$$|m| \sim 10 \eta^2 \delta. \quad (36)$$

There is a tiny repulsive gravitational potential due to the mass term. A freely moving particle near the core experiences an outward proper acceleration:

$$\ddot{r} = -\frac{Gm}{r^2} = \frac{G|m|}{r^2}. \quad (37)$$

Since $|m| \sim 10\eta^2\delta$, the mass term $\sim 10\eta^2\delta/r$. When $r \gg \delta$, the mass term may be negligible. Neglecting the mass term and rescaling the variables r and t , we can rewrite the metric as

$$ds^2 = dt^2 - dr^2 - (1 - 8\mu)r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (38)$$

The metric (38) describes a space with a solid deficit angle.

4 The geodesics

Let us now write down the equations for the geodesics in the metric (19). From

$$\frac{d^2x^\mu}{dp^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{dp} \frac{dx^\lambda}{dp} = 0 \quad (39)$$

we have

$$A(r) \left(\frac{dr}{dp} \right)^2 = \frac{1}{B(r)} - \frac{J^2}{r^2} - E, \quad (40)$$

$$r^2 \frac{d\varphi}{dp} = J, \quad (41)$$

$$\frac{dt}{dp} = \frac{1}{B(r)}. \quad (42)$$

Here J and E are integral constants. J represents the angular momentum of the trajectory and $E^{\frac{1}{2}}$ is the ratio between the proper time along the trajectory and the affine parameter p , i.e. $ds^2 = -E dp^2$.

From equations (40) – (42) the shape of the path, $r = r(\varphi)$, can be found. It is given by

$$\left| \frac{d\varphi}{dr} \right| \equiv \triangle(r) = \frac{1}{r^2} A^{1/2}(r) \left[\frac{1}{JB(r)} - \frac{E}{J^2} - \frac{1}{r^2} \right]^{-1/2} \quad (43)$$

And we can parametrize the trajectory in terms of the distance of closest approach to the core, r_0 , instead of the angular momentum J . Form $\frac{dr}{d\varphi}|_{r_0} = 0$, we have

$$J = r_0 (1/B(r_0) - E)^{1/2}. \quad (44)$$

Then equation (43) reduce to

$$\left| \frac{d\varphi}{dr} \right| \equiv \triangle(r) = \frac{r_0}{r} A^{1/2}(r) \left[\frac{B(r_0)}{B(r)} \frac{1 - EB(r)}{1 - EB(r_0)} - \left(\frac{r_0}{r} \right)^2 \right]^{-1/2} \quad (45)$$

Consider a trajectory starting from a distance $L \gg r_0$, and with velocity V such that $V^2 \gg \mu$. Most of the deflection imprinted upon the trajectory by the gravitational field will occur in the region $r \approx r_0$, where

$$\triangle(r) \approx \frac{r_0}{r} \frac{1}{[1 - (r_0/r)^2]^{1/2}} \left[1 + 4\mu + \frac{Gm}{2r} - \frac{Gm}{4V^2} \frac{1}{r_0} \frac{1}{1 + r_0/r} \right] \quad (46)$$

to the first order of μ and μ/V^2 . Also $E = 1 - V^2$ to this order. The deflection angle is

$$\varepsilon = \pi - \Delta\varphi \quad (47)$$

where

$$\Delta\varphi = 2 \int_{r_0}^{\infty} \Delta(r) dr. \quad (48)$$

Equations (46) – (48) yield

$$\varepsilon = -4\pi\mu - \left(1 - \frac{1}{2V^2}\right) \frac{Gm}{r_0} \quad (49)$$

In our order of approximation, r_0 coincides with the impact parameter of the trajectory. The analogous result to equation (49) for a gauge string is $\varepsilon = -4\pi\mu$ with μ the string energy per unit length. The monopole case differs from that of a gauge string by the dependence of ε upon both the impact parameter and the velocity of the trajectory.

In the case of ultrarelativistic particles ($V \approx 1$), the deflection angle is

$$\varepsilon = -4\pi\mu \left[1 + \frac{Gm}{8\mu\delta} \frac{\delta}{r_0} \right] \quad (50)$$

where $Gm/8\mu\delta = O(1)$. We argue that light rays are deflected by various angles, dependent upon the impact parameter r_0 . If r_0 is a galaxy scale and δ is of the order of grand unification scale, the second term in equation (50) can be neglected. In this case, the deflected angle is independent of the impact parameter. However, if the impact parameter is of the scale of a mini soliton star, the second term in equation (50) cannot be neglected. Such an object can be formed if a global monopole is swallowed by a mini soliton star.

There is a threshold velocity V_0 above which the monopole acts as a convergent lens and below which the monopole acts as a divergent lens. By setting $\varepsilon = 0$ in equation (49) one obtains

$$V_0^2 = \left[2 \left(1 + \frac{4\pi\mu\delta}{Gm} \frac{r_0}{\delta} \right) \right]^{-1}. \quad (51)$$

If r_0 and δ are of the scale of mini soliton star, V_0 is about half the speed of light. When a particle moves at speed V_0 there is no net deflection.

5 discussion

The crucial question now is what is the expected density of global monopoles. Barriola and Vilenkin [17] proposed global monopole-antimonopole annihilation as a possible extremely efficient mechanism for reducing their number. The energy of the pair ($M\overline{M}$) is $E \sim \eta^2 R$, where R is the $M\overline{M}$ distance. The attractive force acting on M and \overline{M} is F and is independent upon the distance. The repulsive gravitational force F_{rep} has maximum effect at distance $R \sim \delta$, where it is

$$F_{\text{rep}} \sim \frac{Gm^2}{\delta^2} \sim 10^3 \mu \eta^2 \quad (52)$$

Because $\mu \sim 10^{-6}$ at the typical grand unification scale 10^{16} GeV, the repulsive force F_{rep} can be neglected according to the large attractive force F . The large attractive force between global monopole and antimonopole suggest that $M\overline{M}$ annihilation is very effective and that the monopole overproduction problem may not exist.

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